p97 Ex4.5 no. 1

1. a) X = no. 6's thrown from two D6

x	0	1	2
$P(X=x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

from considering

	1	2	3	4	5	6
1						✓
2						✓
3						✓
4						✓
5						✓
6	✓	✓	✓	✓	✓	✓✓

b) X = smaller or equal no. when two D6 thrown.

x	1	2	3	4	5	6
$P(X=x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

or $P(X=x) = \frac{2(6-x)+1}{36}$

$P(X=x) = \frac{13-2x}{36} \quad x = 1, 2, \ldots 6.$

c) X = no. heads from 3 fair coins

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(done in p90 Ex4A no. 3)

Ex4.5 no.2

$$P(X=x) = \begin{cases} kx & x=1,2,3,4,5 \\ k(10-x) & x=6,7,8,9 \end{cases}$$

a) $\sum P(X=x) = 1$

So $1 = k(1+2+3+4+5+4+3+2+1)$

$1 = 25k$

$k = \frac{1}{25}$.

x	1	2	3	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{5}{25}$	$\frac{4}{25}$	$\frac{3}{25}$	$\frac{2}{25}$	$\frac{1}{25}$
$x^2P(X=x)$	$\frac{1}{25}$	$\frac{8}{25}$	$\frac{27}{25}$	$\frac{64}{25}$	$\frac{125}{25}$	$\frac{144}{25}$	$\frac{147}{25}$	$\frac{128}{25}$	$\frac{81}{25}$

b) $E(X) = 5$ by symmetry of distribution.

c) $E(X^2) = \sum x^2 P(X=x)$

$= \frac{1}{25} + \frac{8}{25} + \dots \frac{81}{25} = 29$.

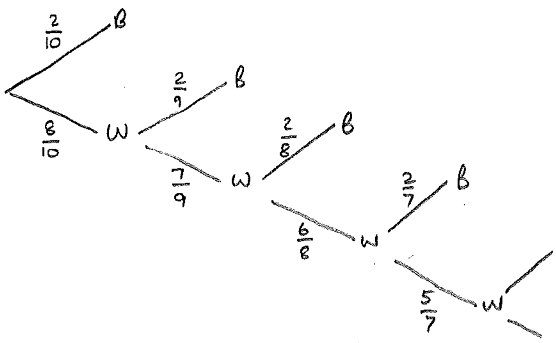
$Var(X) = E(X^2) - E^2(X)$

$= 29 - 5^2$

$= 4$.

Ex4.5 no. 3

withdraw without replacement, 10 discs – 2 black
8 white



etc.

X = no. discs drawn up to and including the first black one

x	1	2	3	4	
$P(X=x)$	$\frac{2}{10}$	$\frac{8}{10}\times\frac{2}{9}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{2}{8}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{2}{7}$	etc.

x	1	2	3	4	5	6	7	8	9
$P(X=x)$	$\frac{18}{90}$	$\frac{16}{90}$	$\frac{14}{90}$	$\frac{12}{90}$	$\frac{10}{90}$	$\frac{8}{90}$	$\frac{6}{90}$	$\frac{4}{90}$	$\frac{2}{90}$

(ie, it seems like $P(X=x) = \frac{20-2x}{90}$)

so $E(X) = \sum x P(X=x)$

$= 1\times\frac{18}{90} + 2\times\frac{16}{90} + \dots + 9\times\frac{2}{90}$

$= \frac{11}{3}$

$E(X^2) = \sum x^2 P(X=x)$

$= 1^2\times\frac{18}{90} + 2^2\times\frac{16}{90} + \dots + 9^2\times\frac{16}{90}$

$= \frac{55}{3}$

so $V(X) = E(X^2) - E^2(X)$

$= \frac{55}{3} - \left(\frac{11}{3}\right)^2$

$= \frac{44}{9}$

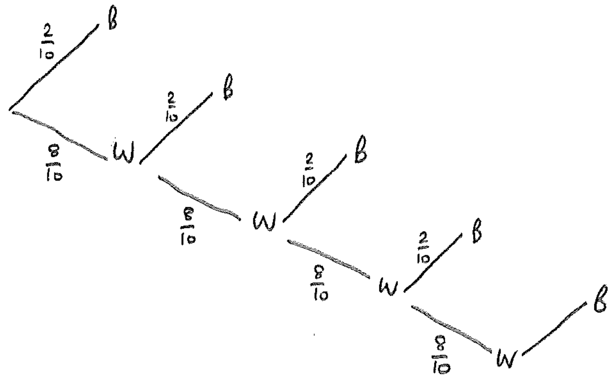
so $\sigma_X = \sqrt{\frac{44}{9}}$

≈ 2.2111 (4dp)

The most likely value of X is one as it has the highest probability (of $\frac{18}{90} = \frac{1}{5}$)

Ex4.5 no. 3 cont.

withdraw <u>with</u> replacement



X = no. discs drawn up to and including the first black one

x	1	2	3	4	etc
$P(X=x)$	$\frac{2}{10}$	$\frac{8}{10} \times \frac{2}{10}$	$\left(\frac{8}{10}\right)^2 \times \frac{2}{10}$	$\left(\frac{8}{10}\right)^3 \times \frac{2}{10}$	

The differences this time are:

- X is no longer restricted to $\{1, 2, \ldots, 9\}$ as it can continue indefinitely
- the probabilities do not change at each stage, due to the replacements

However the most likely outcome remains at 1, as before

extra info

Here, $P(X=x) = \left(\frac{8}{10}\right)^{x-1} \times \frac{2}{10}$

$$= \left(\frac{4}{5}\right)^{x-1} \times \frac{1}{5}$$

This is an example of X having a Geometric Distribution.

Ex4·5 no.4

total = 12 – pay £6
total = 8 – pay £3
die shows 1 – lose £2
} from Gambler's perspective.

	1	2	3	4	5	6
1	-2	-2	-2	-2	-2	-2
2	-2					3
3	-2				3	
4	-2			3		
5	-2		3			
6	-2	3				6

X = amount paid to statistician by gambler

x	6	3	-2	0
$P(X=x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{11}{36}$	$\frac{19}{36}$.

a) $P(X=6) = \frac{1}{36}$

b) $P(X=3) = \frac{5}{36}$

c) $P(X=-2) = \frac{11}{36}$

$$E(X) = \sum x P(X=x)$$
$$= -\frac{1}{36}.$$

So, the gambler would expect to lose £$\frac{1}{36}$ each time the game is played.

Hence, over 100 games the gambler would lose £$\frac{100}{36}$ ≈ £2·78 (2dp)

To make the game unbiased we want $E(X)=0$

$$\Rightarrow \quad 0 = \frac{a}{36} + \frac{15}{36} - \frac{22}{36} + \frac{0}{36}$$

$$\Rightarrow \quad a = 7$$

Ex4.5 no.5

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

let X = difference in scores on two D6

x	0	1	2	3	4	5
$P(X=x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$
	roll again	A wins	A wins	B wins	B wins	B wins

$$\begin{aligned} E(X) &= \sum x P(X=x) \\ &= \frac{1}{36} \times (10+16+18+16+10) \\ &= \frac{70}{36} \\ &= \frac{35}{18}. \end{aligned}$$

a) $$\begin{aligned} P(\text{A wins on first roll}) &= P(X=1)+P(X=2) \\ &= \frac{18}{36} \\ &= \frac{1}{2} \end{aligned}$$

b) $$\begin{aligned} P(\text{A wins on second roll}) &= P(X=0) \times P(\text{A wins}) \\ &= \frac{6}{36} \times \frac{1}{2} \\ &= \frac{1}{6} \times \frac{1}{2} \\ &= \frac{1}{12}. \end{aligned}$$

c) $$\begin{aligned} P(\text{A wins on } r^{th} \text{ roll}) &= P(X=0 \text{ for } r-1 \text{ rolls}) \times P(\text{A wins}) \\ &= \left(\frac{1}{6}\right)^{r-1} \times \frac{1}{2}. \end{aligned}$$

This is a geometric series with $a = \frac{1}{2}$, $r = \frac{1}{6}$, so $S_\infty = \frac{a}{1-r} = \frac{1/2}{1-1/6} = \frac{3/6}{5/6} = \frac{3}{5}$.

Alternatively, on TI-Nspire $\sum_{r=1}^{1000} \left(\frac{1}{6}\right)^{r-1} \times \frac{1}{2}$ will sum the first thousand games' probabilities for A to win.

As A is more likely to win than B, if B stakes £1, then

P(A win) : P(B win)

$\frac{3}{5} : \frac{2}{5}$

3 : 2

1.5 : 1

A should stake £1.50.

Ex4.5 no.6

a) 4 packs. each has $P(\text{red}) = P(\text{blue}) = P(\text{green}) = \frac{1}{3}$

X = number of red cards.

$$P(X=0) = P(\bar{R} \cap \bar{R} \cap \bar{R} \cap \bar{R})$$ where R = draw a red card.

$$= P(\bar{R}) \times P(\bar{R}) \times P(\bar{R}) \times P(\bar{R})$$ as each deck in dependent of other decks.

$$= \left(\tfrac{2}{3}\right) \times \left(\tfrac{2}{3}\right) \times \left(\tfrac{2}{3}\right) \times \left(\tfrac{2}{3}\right)$$

$$= \left(\tfrac{2}{3}\right)^4$$

$$= \frac{16}{81}$$

$$P(X=1) = P(R \cap \bar{R} \cap \bar{R} \cap \bar{R}) \times {}^4C_1$$

$$= P(R) \times P(\bar{R}) \times P(\bar{R}) \times P(\bar{R}) \times 4$$

$$= \tfrac{1}{3} \times \left(\tfrac{2}{3}\right)^3 \times 4$$

$$= \frac{32}{81}$$

$$P(X=2) = P(R \cap R \cap \bar{R} \cap \bar{R}) \times {}^4C_2$$

$$= P(R) \times P(R) \times P(\bar{R}) \times P(\bar{R}) \times {}^4C_2$$

$$= \left(\tfrac{1}{3}\right)^2 \times \left(\tfrac{2}{3}\right)^2 \times 6$$

$$= \frac{24}{81}$$

$$P(X=3) = \left(\tfrac{1}{3}\right)^3 \times \left(\tfrac{2}{3}\right) \times {}^4C_3$$ by similar logic to above

$$= \left(\tfrac{1}{3}\right)^3 \times \left(\tfrac{2}{3}\right) \times 4$$

$$= \frac{8}{81}$$

$$P(X=4) = \left(\tfrac{1}{3}\right)^4$$

$$= \frac{1}{81}$$

So

x	0	1	2	3	4
$P(X=x)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

6b) let Y = net winnings per game.

x	0	1	2	3	4
y	-2	-2	1	3	8
$P(Y=y)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

(X as defined in part (a))

$$\therefore E(Y) = \sum y P(Y=y)$$

$$= -2 \times \frac{16}{81} + -2 \times \frac{32}{81} + 1 \times \frac{24}{81} + 3 \times \frac{8}{81} + 8 \times \frac{1}{81}$$

$$= \frac{1}{81}(-32 - 64 + 24 + 24 + 8)$$

$$= -\frac{40}{81}$$

$$\approx -0.4938 \text{ (4dp)}$$

$$\approx -0.50 \text{ (2dp)}$$

i.e. expected winnings $= -£0.50$ i.e. a loss of 50p. per game, in the long run.

Ex4.5 no. 7

if roll 6, roll again & sum
if roll ≤ 5, stick.

X = score obtained

x	1	2	3	4	5	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$.

or $P(X=x) = \begin{cases} \frac{1}{6} & x = 1,2,3,4,5 \\ \frac{1}{36} & x = 7,8,9,10,11,12 \end{cases}$

$$E(X) = \sum x P(X=x)$$
$$= \underline{\underline{\frac{49}{12}}}.$$

Consider now playing the above game twice, one after the other, to give two successive scores for X,

$P(\text{sum of two successive scores} \geq 8) = 1 - P(\text{sum of successive scores} \leq 7)$

$$= 1 - \big(P(X=1)P(X\leq 6) + P(X=2)P(X\leq 5) + P(X=3)P(X\leq 4) + P(X=4)P(X\leq 3) + P(X=5)P(X\leq 2)\big)$$

$$= 1 - \left(\frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot\frac{4}{6} + \frac{1}{6}\cdot\frac{3}{6} + \frac{1}{6}\cdot\frac{2}{6}\right)$$

$$= 1 - \frac{5+5+4+3+2}{36}$$

$$= 1 - \frac{19}{36}$$

$$= \underline{\underline{\frac{17}{36}}}. \quad \text{as required.}$$

$$P(\text{first game score} \geq 7 \mid \text{sum of both game scores} \geq 8) = \frac{P(\text{first game score} \geq 7 \text{ and sum of game scores} \geq 8)}{P(\text{sum of game scores} \geq 8)}$$

$$= \frac{P(\text{second game score} \geq 1) \cdot P(\text{first game score} \geq 7)}{P(\text{sum of game scores} \geq 8)}$$

$$= \frac{1 \times \frac{1}{6}}{17/36}$$

$$= \frac{6/36}{17/36}$$

$$= \underline{\underline{\frac{6}{17}}}.$$

Ex4.5 no.8.

x	0	1	2	3	4	5
$P(X=x)$	a	a	a	b	b	b

we have $P(X\geq 2) = 3P(X<2)$ so $P(X\geq 2) = a+b+b+b$ $= a+3b$ and $P(X<2) = a+a$ $= 2a$.

$\Rightarrow a+3b = 3(2a)$

$a+3b = 6a$

$3b = 5a$

also $\sum P(X=x) = 1 \Rightarrow 3a+3b=1$

$a+b = \frac{1}{3}$.

$$\left.\begin{array}{l} 3b = 5a \\ 3a+3b=1 \end{array}\right\}$$

$3a+3b=1$

$3a+5a=1$

$a = \frac{1}{8}$

$\Rightarrow b = \frac{1}{3} - \frac{1}{8}$

$b = \frac{8}{24} - \frac{3}{24}$

$b = \frac{5}{24}$

so $a=\frac{1}{8}$, $b=\frac{5}{24}$.

ii) $E(X) = \sum xP(X=x)$

$= 3a+12b$

$= \frac{3}{8} + \frac{60}{24}$

$= \frac{69}{24}$

$= \frac{23}{8}$. as required.

$E(X^2) = \sum x^2 P(X=x)$

$= 5a + (3^2+4^2+5^2)b$

$= 5a+50b$

$= \frac{5}{8} + \frac{250}{24}$

$= \frac{15}{24} + \frac{250}{24}$

$= \frac{265}{24}$

so $Var(X) = E(X^2) - E^2(X)$

$= \frac{265}{24} - \left(\frac{23}{8}\right)^2$

$= \frac{533}{192}$

≈ 2.7760 (4dp)

iii) let $Y = X_1 + X_2$

X_1 \ X_2	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10

$P(Y>7) = P(Y\geq 8)$

$= P(X_1=3)P(X_2=5) + P(X_1=4)P(X_2\geq 4) + P(X_1=5)P(X_2\geq 3)$

$= (b)(b) + (b)(2b) + (b)(3b)$

$= b^2 + 2b^2 + 3b^2$

$= 6b^2$

$= 6\times\left(\frac{5}{24}\right)^2$

$= \frac{25}{96}$

≈ 0.2604 (4dp)