p148 Ex 7.7 no. 1

1. X = petrol consumption, in 1000's of litres.

$$f(x) = \begin{cases} ax^2(b-x) & 0<x<1 \\ 0 & \text{otherwise,} \end{cases}$$

so valid pdf and mean consumption = 600 litres
= 0.6 thousand litres.

so $\int_0^1 f(x)\,dx = 1$ and $\int_0^1 x f(x)\,dx = 0.6$.

$$\int_0^1 ax^2(b-x)\,dx = 1$$

$$a\int_0^1 (bx^2 - x^3)\,dx = 1$$

$$\left[\tfrac{1}{3}bx^3 - \tfrac{1}{4}x^4\right]_0^1 = \tfrac{1}{a}$$

$$\left(\tfrac{1}{3}b - \tfrac{1}{4}\right) - (0-0) = \tfrac{1}{a}$$

$$\tfrac{1}{3}b - \tfrac{1}{4} = \tfrac{1}{a} \quad \times 12a$$

$$4ab - 3a = 12$$

$$a(4b-3) = 12$$

$$a = \frac{12}{4b-3}$$

$$\int_0^1 ax^3(b-x)\,dx = 0.6$$

$$a\int_0^1 (bx^3 - x^4)\,dx = \tfrac{6}{10}$$

$$\left[\tfrac{1}{4}bx^4 - \tfrac{1}{5}x^5\right]_0^1 = \frac{3}{5a}$$

$$\left(\tfrac{1}{4}b - \tfrac{1}{5}\right) - (0-0) = \frac{3}{5a}$$

$$\tfrac{1}{4}b - \tfrac{1}{5} = \frac{3}{5a} \quad \times 20a$$

$$5ab - 4a = 12.$$

$$a(5b-4) = 12.$$

$$a = \frac{12}{5b-4}.$$

$$\frac{12}{4b-3} = \frac{12}{5b-4}$$

$$4b-3 = 5b-4$$

$$1 = b.$$

so $b=1 \Rightarrow a = \dfrac{12}{4\times1-3} = 12.$

so $\underline{a=12,\ b=1}$

$$P(\text{consumption exceeds 900 litres}) = P(X>0.9)$$

$$= \int_{0.9}^1 12x^2(1-x)\,dx$$

$$= \underline{\underline{0.0523}}$$, by TI-Nspires.

Ex 7.7 no. 2

$f(x) = kx^2(3-x) \qquad 0 < x < 3$

a) valid if $\int_0^3 f(x)\,dx = 1$

$$\int_0^3 kx^2(3-x)\,dx = 1$$
$$k\int_0^3 (3x^2 - x^3)\,dx = 1$$
$$\left[x^3 - \tfrac{1}{4}x^4\right]_0^3 = \frac{1}{k}$$
$$\left(3^3 - \tfrac{1}{4}3^4\right) - 0 = \frac{1}{k}$$
$$27 - \frac{81}{4} = \frac{1}{k}$$
$$\frac{108-81}{4} = \frac{1}{k}$$
$$\frac{27}{4} = \frac{1}{k}$$
$$\underline{\underline{k = \frac{4}{27}}}$$

b) mean $= \int_0^3 x.f(x)\,dx$

$$= \int_0^3 \frac{4}{27}x^3(3-x)\,dx$$
$$= \frac{4}{27}\int_0^3 (3x^3 - x^4)\,dx$$
$$= \frac{4}{27}\left[\tfrac{3}{4}x^4 - \tfrac{1}{5}x^5\right]_0^3$$
$$= \frac{4}{27}\left[\frac{3}{4}.81 - \frac{1}{5}.243 - 0\right]$$
$$= \frac{4}{27}\left[\frac{243}{4} - \frac{243}{5}\right]$$
$$= \frac{4.243}{27}\left[\frac{1}{4} - \frac{1}{5}\right]$$
$$= \frac{4.3^5}{3^3}\left[\frac{1}{20}\right]$$
$$= 4.3^2.\frac{1}{20}$$
$$= \underline{\underline{\frac{9}{5}}}$$

$E(X^2) = \int_0^3 x^2 f(x)\,dx$

$$= \frac{4}{27}\int_0^3 (3x^4 - x^5)\,dx$$
$$= \frac{4}{27}\left[\tfrac{3}{5}x^5 - \tfrac{1}{6}x^6\right]_0^3$$
$$= \frac{4}{3^3}\left[\frac{3}{5}.3^5 - \frac{1}{6}.3^6 - 0\right]$$
$$= \frac{4}{3^3}\left[\frac{3^6}{5} - \frac{3^6}{6}\right]$$
$$= \frac{4.3^6}{3^3}\left[\frac{1}{5} - \frac{1}{6}\right]$$
$$= 4.3^3.\frac{1}{30}.$$
$$= \frac{2}{15}.3.3^2$$
$$= \frac{18}{5}$$

so $Var(X) = E(X^2) - E^2(X)$

$$= \frac{18}{5} - \left(\frac{9}{5}\right)^2$$
$$= \frac{90}{25} - \frac{81}{25}$$
$$= \underline{\underline{\frac{9}{25}}}$$

$\mu = \frac{9}{5} \quad s^2 = \frac{9}{25} \Rightarrow s = \frac{3}{5}.$

$\mu + 2s = \frac{9}{5} + \frac{6}{5}$

$= \frac{15}{5} = 3$ ✓

no. 2 cont.

c) $P(|x-\mu| > 2s)$ $\qquad\qquad \mu = \frac{9}{5}, \; s = \frac{3}{5}$

$= 1 - P(\mu - 2s < x < \mu + 2s)$ $\qquad\qquad \mu - 2s = \frac{9}{5} - \frac{6}{5} = \frac{3}{5}$

$= 1 - P\left(\frac{3}{5} < x < 3\right)$

$= P\left(0 < x < \frac{3}{5}\right)$

$= \int_0^{3/5} f(x)\,dx$

$= \int_0^{3/5} \frac{4}{27} x^2 (3-x)\,dx$

$= \frac{4}{27} \int_0^{3/5} (3x^2 - x^3)\,dx$

$= \frac{4}{27} \left[x^3 - \frac{1}{4} x^4 \right]_0^{3/5}$

$= \frac{4}{27} \left[\left(\frac{3}{5}\right)^3 - \frac{1}{4}\left(\frac{3}{5}\right)^4 - 0 \right]$

$= \frac{4}{27} \cdot \left(\frac{3}{5}\right)^3 \left[1 - \frac{1}{4} \cdot \frac{3}{5} \right]$

$= \frac{4}{27} \cdot \frac{27}{125} \cdot \left(1 - \frac{3}{20}\right)$

$= \frac{4}{125} \cdot \frac{17}{20}$

$= \frac{1}{125} \cdot \frac{17}{5}$

$= \frac{17}{625}$

$= \underline{\underline{0.0272}}$

Ex 7.7 no. 3.

$$f(x) = \begin{cases} x(x-1)(x-2) & 0<x<1 \\ a & 1<x<3 \\ 0 & \text{otherwise} \end{cases}$$

a) $\int_0^3 f(x)\,dx = 1$ if valid pdf.

$$\int_0^1 x(x-1)(x-2)\,dx + \int_1^3 a\,.dx = 1$$

$$\int_0^1 x(x^2-3x+2)\,dx + \left[ax\right]_1^3 = 1$$

$$\int_0^1 (x^3-3x^2+2x)\,dx + (3a-a) = 1$$

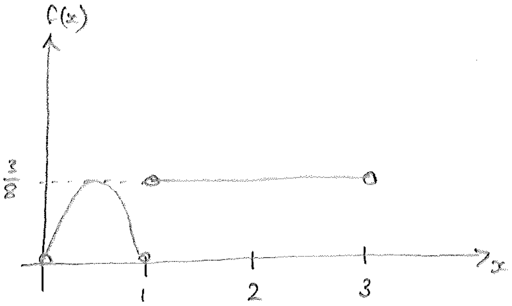
$$\left[\tfrac{1}{4}x^4 - x^3 + x^2\right]_0^1 + 2a = 1$$

$$\left(\tfrac{1}{4}-1+1\right)-(0-0+0) + 2a = 1$$

$$2a = \tfrac{3}{4}$$

$$a = \tfrac{3}{8}.$$

b)



c) $E(X) = \int_0^3 x.f(x)\,dx$

$$= \int_0^1 (x^4-3x^3+2x^2)\,dx + \int_1^3 ax\,.dx$$

$$= \left[\tfrac{1}{5}x^5 - \tfrac{3}{4}x^4 + \tfrac{2}{3}x^3\right]_0^1 + \left[\tfrac{1}{2}ax^2\right]_1^3$$

$$= \left(\tfrac{1}{5} - \tfrac{3}{4} + \tfrac{2}{3}\right) - (0) + \tfrac{9}{2}a - \tfrac{1}{2}a$$

$$= \frac{12-45+40}{60} + 4a$$

$$= \frac{7}{60} + 4.\frac{3}{8}$$

$$= \frac{7}{60} + \frac{3}{2}$$

$$= \frac{7}{60} + \frac{90}{60}$$

$$= \frac{97}{60}$$

$P(X < E(X))$

$$= P\left(X < \tfrac{97}{60}\right)$$

$$= \int_0^1 f(x)\,dx + \int_1^{97/60} f(x)\,dx$$

$$= \tfrac{1}{4} + \left[ax\right]_1^{97/60}$$

$$= \tfrac{1}{4} + \tfrac{37}{60}a$$

$$= \tfrac{1}{4} + \tfrac{37}{60}\cdot\tfrac{3}{8}$$

$$= \tfrac{1}{4} + \tfrac{37}{20}\cdot\tfrac{1}{8}$$

$$= \tfrac{40}{160} + \tfrac{37}{160}$$

$$= \tfrac{77}{160}.$$

now, for median, $P(X < \text{median}) = \frac{1}{2}$

as $\frac{77}{160} < \frac{1}{2}$, the mean is 'to the left' of the median ie. mean < median.

Ex 7.7 no. 4.

X = amount in inedibles, in kg, in a 5kg bag.

$$f(x) = \begin{cases} k(x-1)(3-x) & 1<x<3 \\ 0 & \text{otherwise} \end{cases}$$

a) valid pdf if $\int_1^3 f(x)\,dx = 1$

$$k\int_1^3 (x-1)(3-x)\,dx = 1$$

$$\int_1^3 (-x^2+4x-3)\,dx = \frac{1}{k}$$

$$\left[-\frac{1}{3}x^3+2x^2-3x\right]_1^3 = \frac{1}{k}$$

$$\left(-\frac{1}{3}\cdot 27 + 2\cdot 9 - 3\cdot 3\right) - \left(-\frac{1}{3}+2-3\right) = \frac{1}{k}$$

$$(-9+18-9) - \left(-\frac{1}{3}-1\right) = \frac{1}{k}$$

$$0 + \frac{4}{3} = \frac{1}{k}$$

$$\underline{k = \frac{3}{4}}$$

b) $E(X) = \int_1^3 x . f(x) . dx$

$= \int_1^3 x \frac{3}{4}(x-1)(3-x)\,dx$

$= 2$ by TI-Nspire

$E(X^2) = \int_1^3 x^2 f(x)\,dx$

$= 4.2$ by TI-Nspire

so $Var(X) = E(X^2) - E^2(X)$

$= 4.2 - 4^2$

$= 0.2$

$= \frac{1}{5}$

$\therefore$ $\underline{E(X) = 2,\ Var(X) = \frac{1}{5}}$.

c) $P(X > 2.5) = \int_{2.5}^3 f(x)dx$

$= \underline{\frac{5}{32}}$ by TI-Nspire.

Ex 7.7 no. 5.

X = daily demand in gallons.

$$f(x) = \begin{cases} kx^2(10-x) & 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a) valid pdf if $\int_0^{10} f(x)\,dx = 1$

$$k\int_0^{10} x^2(10-x)\,dx = 1$$

$$\int_0^{10} (10x^2 - x^3)\,dx = \frac{1}{k}$$

$$\left[\frac{10}{3}x^3 - \frac{1}{4}x^4\right]_0^{10} = \frac{1}{k}$$

$$\left(\frac{10.10^3}{3} - \frac{1}{4}.10^4\right) - 0 = \frac{1}{k}$$

$$10^4\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{k}$$

$$10^4.\left(\frac{4}{12} - \frac{3}{12}\right) = \frac{1}{k}$$

$$\frac{10^4}{12} = \frac{1}{k}$$

$$k = \frac{12}{10^4}$$

$$\underline{k = 0.00012}$$

b) $E(X) = \int_0^{10} x\, f(x)\,dx$

$= 6$ by TI-Nspire

$E(X^2) = \int_0^{10} x^2 f(x)\,dx$

$= 40$ by TI-Nspire

$Var(X) = E(X^2) - E^2(X)$

$= 40 - 6^2$

$= 4$

$\Rightarrow$ st. dev $= \sqrt{4} = 2$

So mean $= 6$, st. dev $= 2$

c) $f(x)$ is maximised when $f'(x) = 0$

$f(x) = kx^2(10-x)$

$= k.10.x^2 - kx^3$

$f'(x) = 20kx - 3kx^2$

$f'(x) = 0$ when $20kx - 3kx^2 = 0$

$kx(20 - 3x) = 0$

$x = 0$ or $\frac{20}{3}$

check it's a maximum:

$f''(x) = 20k - 6kx$

$f''\left(\frac{20}{3}\right) = 20k - 6.k.\frac{20}{3}$

$= 20k(1-2)$

$= -20k$

< 0

$\therefore$ max @ $\underline{\underline{x = \frac{20}{3}}}$

This value is the <u>mode</u>

d) mean $= 6$

mode $= \frac{20}{3}$

$2(\text{median} - \text{mean}) = \text{mode} - \text{median}$

$2(m - 6) = \frac{20}{3} - m$

$2m - 12 = \frac{20}{3} - m$

$3m = 12 + \frac{20}{3}$

$3m = \frac{56}{3}$

$\underline{\text{median} = \frac{56}{9}}$

if it is median, then $\int_0^{\text{median}} f(x)\,dx = \frac{1}{2}$

now $\int_0^{56/9} f(x)\,dx = 0.5139197\ldots$ by TI-Nspire

≈ 0.5

Hence answer is approximately correct.

e) Mean daily sales if only able to sell maximum of 8 gallons. $= \int_0^8 x f(x)\,dx + \int_8^{10} 8.f(x)\,dx$

$= 4.42368\ldots + 1.4464\ldots$

$= 5.87008\ldots$

$\approx \underline{\underline{5.87}}$ (3sf)

Ex 7.7 no. 6

T = journey time to work by car, in hours.

$$f(t) = \begin{cases} 10ct^2 & 0 \le t < 0.6 \\ 9c(1-t) & 0.6 \le t \le 1.0 \\ 0 & \text{otherwise.} \end{cases}$$

a) $\int_0^1 f(t)\,dt = 1$ for valid pdf.

on TI-Nspire, define $f(t) = \begin{cases} 10ct^2, & 0 \le t < 0.6 \\ 9c(1-t), & 0.6 \le t \le 1.0 \end{cases}$

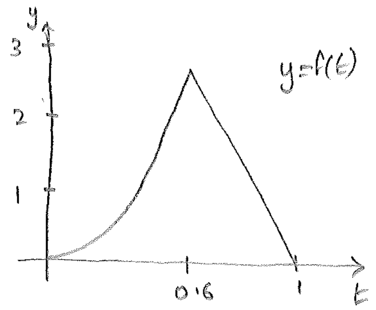
and nSolve $\left(\int_0^1 f(t)\,dt = 1, c \right)$

to get $c = 0.694444\ldots$

then Ans ▸ approxFraction (5E-8) gives $\frac{25}{36}$.

then define $c := \frac{25}{36}$.

and go to Graph page to draw $f1(x) = f(x)$



b) most likely time = mode
= where maximum is
= 0.6 hr (36 mins)

c) i) $P\left(T > \frac{48}{60}\right) = \int_{\frac{48}{60}}^1 f(t)\,dt$

$= 0.125$ by TI-Nspire

$= \frac{1}{8}$.

ii) $P\left(\frac{24}{60} < T < \frac{48}{60}\right) = \int_{24/60}^{48/60} f(t)\,dt$

$= 0.726852\ldots$ by TI-Nspire

≈ 0.727 (3dp)

Ex 7.7 no. 7

X = distance from centre of target

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

target radius = 5cm → 2pts.
bull radius = 2cm → 5pts

a) $P(\text{hit the bull}) = P(X<2)$

$$= \int_0^2 f(x)\,dx$$
$$= \int_0^2 \frac{1}{4}e^{-\frac{x}{4}}\,dx$$
$$= \frac{1}{4}\int_0^2 e^{-\frac{x}{4}}\,dx$$
$$= \frac{1}{4}\left[-4e^{-\frac{x}{4}}\right]_0^2$$
$$= \left[-e^{-x/4}\right]_0^2$$
$$= -e^{-2/4} - (-e^0)$$
$$= -e^{-\frac{1}{2}} + 1$$
$$= 1 - \frac{1}{\sqrt{e}}$$
$$\approx 0.393469..$$
$$\approx \underline{0.3935} \text{ (4dp)}$$

b) $P(\text{miss target}) = P(X>5)$

$$= \int_5^\infty f(x)\,dx$$
$$= \left[-e^{-x/4}\right]_5^\infty$$
$$= -e^{-\infty} - (-e^{-5/4})$$
$$= e^{-5/4}$$
$$\approx 0.286505...$$
$$\approx \underline{0.2865} \text{ (4dp)}$$

c)

score	5pts	2pts	0pts.
P(score)	$1-\frac{1}{\sqrt{e}}$		$e^{-5/4}$
	0.3934	0.320026	0.2865

so $E(\text{score}) = 5\times0.3934 + 2\times0.320026 + 0$

$$= 2.6074...$$
$$\approx \underline{2.6} \text{ (2sf)}$$

Ex7.7 no.8.

$$f(x) = \frac{2}{3}\cos\left(x - \frac{\pi}{6}\right) \qquad 0 < x < \frac{2\pi}{3}$$

a) valid pdf if $\int_0^{2\pi/3} f(x)\,dx = 1$

$$\text{so} \quad \int_0^{\frac{2\pi}{3}} f(x)\,dx = \int_0^{\frac{2\pi}{3}} \frac{2}{3}\cos\left(x - \frac{\pi}{6}\right)dx$$

$$= \frac{2}{3}\int_0^{\frac{2\pi}{3}} \cos\left(x - \frac{\pi}{6}\right)dx$$

$$= \frac{2}{3}\left[\sin\left(x - \frac{\pi}{6}\right)\right]_0^{2\pi/3}$$

$$= \frac{2}{3}\left\{\sin\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) - \sin\left(0 - \frac{\pi}{6}\right)\right\}$$

$$= \frac{2}{3}\left\{\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{6}\right)\right\}$$

$$= \frac{2}{3}\left\{1 + \sin\left(\frac{\pi}{6}\right)\right\}$$

$$= \frac{2}{3}\left\{1 + \frac{1}{2}\right\}$$

$$= \frac{2}{3} \cdot \frac{3}{2}$$

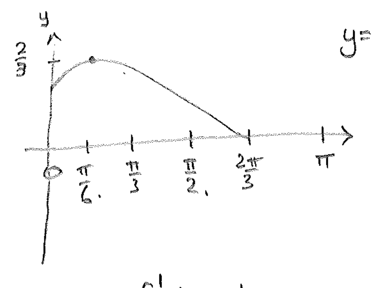
$$= 1$$

b) Mode will be when f(x) at its maximum

now $y = \cos x$ has maximum at $x = 0, 2\pi$

so $x - \frac{\pi}{6} = 0$ or 2π

$x = \frac{\pi}{6}$ or $\cancel{\frac{13}{6}}$.



c) $P(X<1) = \int_0^1 f(x)\,dx$

$= 0.639056$ from TI-Nspire

$\approx \underline{0.639}$ (3dp)

Ex 7·7 no. 9

$X =$ life of electrical component

$f(x) = \frac{100}{x^2} \qquad x > 100$

a) Median, m, when $\int_{100}^{m} f(x)\,dx = \frac{1}{2}$.

$$\int_{100}^{m} \frac{100}{x^2} \cdot dx = \frac{1}{2}$$

$$\left[-100x^{-1} \right]_{100}^{m} = \frac{1}{2}$$

$$\left[-\frac{100}{x} \right]_{100}^{m} = \frac{1}{2}$$

$$-\frac{100}{m} - \left(-\frac{100}{100} \right) = \frac{1}{2}$$

$$1 - \frac{100}{m} = \frac{1}{2} \qquad) \times (2m)$$

$$2m - 200 = m$$

$$m = 200$$

median is 200 hours.

b) $P(X > 250) = \int_{250}^{\infty} f(x)\,dx$

$$= \left[-\frac{100}{x} \right]_{250}^{\infty}$$

$$= -\frac{100}{\infty} - \left(-\frac{100}{250} \right)$$

$$= \frac{100}{250}$$

$$= \frac{10}{25}$$

$$= \frac{2}{5}.$$