p142 Ex 7C no. 1

$f(x) = 6x(1-x) \quad 0<x<1$

$E(X) = \int_0^1 x f(x)\,dx$

$E(X^2) = \int_0^1 x^2 f(x)\,dx$

$Var(X) = E(X^2) - E^2(X)$

so $E(X) = \int_0^1 x \cdot 6x(1-x)\,dx$

$= \int_0^1 (6x^2 - 6x^3)\,dx$

$= \left[2x^3 - \frac{6}{4}x^4\right]_0^1$

$= \left(2 - \frac{6}{4}\right) - (0-0)$

$= 2 - \frac{3}{2}$

$= \frac{1}{2}$.

$E(X^2) = \int_0^1 x^2 \cdot 6x(1-x)\,dx$

$= \int_0^1 (6x^3 - 6x^4)\,dx$

$= \left[\frac{6}{4}x^4 - \frac{6}{5}x^5\right]_0^1$

$= \left(\frac{6}{4} - \frac{6}{5}\right) - (0-0)$

$= 6\left(\frac{1}{4} - \frac{1}{5}\right)$

$= \frac{6}{20}$

$= \frac{3}{10}$.

so $Var(X) = E(X^2) - E^2(X)$

$= \frac{3}{10} - \left(\frac{1}{2}\right)^2$

$= \frac{3}{10} - \frac{1}{4}$

$= \frac{6}{20} - \frac{5}{20}$

$= \frac{1}{20}$.

Ex 7C no. 2

2. R = length of radii

$f(r) = \frac{1}{4} r \quad 1 < r < 3$ (small circles!!)

$$E(R) = \int_1^3 r . f(r) \, dr$$

$$= \int_1^3 \frac{1}{4} r^2 . dr$$

$$= \left[\frac{1}{12} r^3 \right]_1^3$$

$$= \frac{3^3 - 1^3}{12}$$

$$= \frac{27 - 1}{12}$$

$$= \frac{26}{12}$$

$$= \underline{\underline{\frac{13}{6}}}$$

$$E(R^2) = \int_1^3 r^2 . f(r) \, dr$$

$$= \int_1^3 \frac{1}{4} r^3 . dr$$

$$= \left[\frac{1}{16} r^4 \right]_1^3$$

$$= \frac{3^4 - 1^4}{16}$$

$$= \frac{81 - 1}{16}$$

$$= \frac{80}{16}$$

$$= 5$$

so $Var(R) = E(R^2) - E^2(R)$

$$= 5 - \left(\frac{13}{6}\right)^2$$

$$= 5 - \frac{169}{36}$$

$$= \frac{180}{36} - \frac{169}{36}.$$

$$= \underline{\underline{\frac{11}{36}.}}$$

Ex 7C no. 3.

X = proportion of cloud cover

$f(x) = 12x(1-x)^2 \quad 0<x<1$

$$E(X) = \int_0^1 x \cdot f(x)\,dx$$
$$= \int_0^1 12x^2(1-x)^2 \cdot dx$$
$$= 12\int_0^1 x^2(1-2x+x^2)\,dx$$
$$= 12\int_0^1 (x^2-2x^3+x^4)\,dx$$
$$= 12\left[\tfrac{1}{3}x^3 - \tfrac{2}{4}x^4 + \tfrac{1}{5}x^5\right]_0^1$$
$$= 12\left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} - 0\right)$$
$$= 12\left(\frac{10-15+6}{30}\right)$$
$$= 12 \cdot \frac{1}{30}$$
$$= \frac{2}{5}.$$

$$E(X^2) = 12\int_0^1 (x^3-2x^4+x^5)\,dx$$
$$= 12\left[\tfrac{1}{4}x^4 - \tfrac{2}{5}x^5 + \tfrac{1}{6}x^6\right]_0^1$$
$$= 12\left(\tfrac{1}{4} - \tfrac{2}{5} + \tfrac{1}{6} - 0\right)$$
$$= 12\left(\frac{15-24+10}{60}\right)$$
$$= 12 \cdot \frac{1}{60}$$
$$= \frac{1}{5}$$

so $Var(X) = E(X^2) - E^2(X)$
$$= \frac{1}{5} - \left(\frac{2}{5}\right)^2$$
$$= \frac{5}{25} - \frac{4}{25}$$
$$= \frac{1}{25}$$

Ex 7C no. 4

$f(x) = ke^{-x} \quad x > 0$

valid if $\int_0^\infty f(x)\,dx = 1$

so $\int_0^\infty k.e^{-x}.dx = 1$

$k \int_0^\infty e^{-x}\,dx = 1$

$\left[-e^{-x}\right]_0^\infty = \frac{1}{k}$

$(-0-(-1)) = \frac{1}{k}$

$\underline{\underline{k = 1}}.$

so $f(x) = e^{-x} \quad x > 0$

$E(X) = \int_0^\infty x.e^{-x}.dx$

$\underline{\underline{= 1}}$ by Nspire CAS (to avoid daily integration by parts).

$E(X^2) = 2$ by Nspire CAS

so $Var(X) = E(X^2) - E^2(X)$

$= 2 - 1^2$

$= 2 - 1$

$\underline{\underline{= 1}}.$

Ex 7C no. 5.

$f(x) = Ax(6-x)^2 \quad 0 < x < 6.$

so $\int_0^6 f(x)\,dx = 1$

$\int_0^6 Ax(6-x)^2\,dx = 1$

$A\int_0^6 x(36-12x+x^2)\,dx = 1$

$\int_0^6 (36x-12x^2+x^3)\,dx = \frac{1}{A}$

$\left[18x^2-4x^3+\frac{1}{4}x^4\right]_0^6 = \frac{1}{A}$

$(18\times6^2-4\times6^3+\frac{1}{4}\times6^4)-0 = \frac{1}{A}$

$6^2\times(18-4\times6+\frac{1}{4}\times6^2) = \frac{1}{A}$

$36\times(18-24+9) = \frac{1}{A}$

$36\times3 = \frac{1}{A}$

$A = \frac{1}{108}$

$E(X) = \int_0^6 x.\frac{1}{108}.x.(6-x)^2.dx$

$= \frac{1}{108}\int_0^6 x^2(6-x)^2\,dx$

$= \frac{1}{108}\int_0^6 (36x^2-12x^3+x^4)\,dx$

$= \frac{1}{108}\left[12x^3-3x^4+\frac{1}{5}x^5\right]_0^6$

$= \frac{1}{108}\left((12\times6^3-3\times6^4+\frac{1}{5}\times6^5)-0\right)$

$= \frac{1}{108}.6^3.(12-3\times6+\frac{1}{5}\times6^2)$

$= \frac{1}{108}.216.(12-18+7\frac{1}{5})$

$= \frac{216}{108}.\frac{6}{5}$

$= \frac{216}{18}.\frac{1}{5}$

$= \frac{72}{6}.\frac{1}{5}$

$= \frac{12}{1}.\frac{1}{5}$

$= \frac{12}{5}.$

$E(X^2) = \frac{1}{108}\int_0^6 (36x^3-12x^4+x^5)\,dx$

$= \frac{1}{108}\left[9x^4-\frac{12}{5}x^5+\frac{1}{6}x^6\right]_0^6$

$= \frac{1}{108}(9.6^4-\frac{12}{5}.6^5+\frac{1}{6}.6^6-0)$

$= \frac{1}{108}.6^4.(9-\frac{12}{5}\times6+\frac{1}{6}\times6^2)$

$= \frac{216}{108}.6.(9-\frac{72}{5}+6)$

$= \frac{216}{18}.(15-14\frac{2}{5})$

$= 12.\frac{3}{5}$

$= \frac{36}{5}$

so $Var(X) = E(X^2)-E^2(X)$

$= \frac{36}{5}-\left(\frac{12}{5}\right)^2$

$= \frac{180}{25}-\frac{144}{25}$

$= \frac{36}{25}.$